Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 19, 2002*
Revised: *May 1, 2003*
Accepted: *May 2, 2003*

Two AdS_2 branes in the euclidean AdS_3

Sylvain Ribault

Centre de Physique Théorique, Ecole Polytechnique
91128 Palaiseau, France
E-mail: ribault@cpht.polytechnique.fr

Abstract: We compute the density of open strings stretching between AdS_2 branes in the euclidean AdS_3. This is done by solving the factorization constraint of a degenerate boundary field, and the result is checked by a Cardy-type computation. We mention applications to branes in the minkowskian AdS_3 and its cigar coset.

Keywords: D-branes, Conformal Field Models in String Theory.

JHEP05(2003)003

http://jhep.sissa.it/archive/papers/jhep052003003/jhep052003003.pdf

The AdS_2 D-branes in AdS_3 have received some attention after the work of Bachas and Petropoulos [1]. After some semi-classical studies [2, 3], important progress was made with the determination of exact boundary states [4, 5] and open-string spectra [4] for the AdS_2 branes in the euclidean AdS_3. This progress relies on the understanding of closed string theory in AdS_3 and its euclidean version H_3^+ [6, 7].

However, the determination of the boundary states was done by solving the factorization constraint of only one degenerate bulk field. Their consistency is thus far from being proven [8]. Another problem appeared in [4], where it was argued that a consistent density of state for open strings stretched between two different AdS_2 branes could not be found by solving the boundary factorization constraints. It would be very strange that each individual AdS_2 brane would be physical, but that one could not consistently stretch open strings between them.

By examining carefully the analyticity properties of the density of state, I will argue that it is in fact possible to find a solution to the boundary factorization constraint for two different AdS_2 branes. The result can be found in eqs. (6) and (7) (written in terms of reflection amplitudes; I will explain their relation with open-string densities).

This result is not only reassuring for the consistency of the AdS_2 branes in H_3^+, but also has implications for branes in AdS_3 and the cigar $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. Indeed, strings stretching between two opposite AdS_2 branes are related by spectral flow to strings with winding one-half. Thus, our result (6) gives the density of long strings with odd winding number living on a single AdS_2 brane in AdS_3. As predicted in [3], this differs from the density of long strings with even winding number. Moreover, the AdS_2 branes descend to D1 branes in the cigar, which have open string modes with half-integer winding. Their density is also given by eq. (6), see [9]. This was in fact the original motivation for this work.

The plan is as follows. First the density of states $N(j|r_1,r_2)$ and reflection amplitude $R(j|r_1,r_2)$ for open strings between two AdS_2 branes are introduced. The constraint on $R(j|r_1,r_2)$ is written (1) and the solution given explicitly (3)-(7). These formulas, and the sense in which they solve (1), are then made precise by a discussion of the choice of the branch cuts in the function $S'_k(x)$ (5). These results on the open string spectrum are checked by a Cardy-like computation using the boundary state (12). The note ends with a brief explanation of why the spectrum of open strings between two different AdS_2 branes was not found in [4].

Now let me define precisely the quantities I want to compute and the notations I will use. The spectrum of strings in H_3^+ is made of continuous representations of $SL_2(\mathbb{R})$ with a spin $j=-\frac{1}{2}+iP\in -\frac{1}{2}+i\mathbb{R}$ and a Casimir $-j(j+1)$. All these states behave asymptotically as plane waves. If we write $ds^2=d\psi^2+\cosh^2\psi(e^{2\chi}d\nu^2+d\chi^2)$ the metric of H_3^+, then we will be interested in AdS_2 branes of equations $\psi=r$ for some real constant r.

The spectrum of strings stretching between two such branes $\psi=r_1$ and $\psi=r_2$ can be described by a density of states $N(j|r_1,r_2)=N(P|r_1,r_2)$. This is linked to another physical quantity, the reflection amplitude $R(j|r_1,r_2)$ such that $N(j|r_1,r_2)=\frac{1}{2\pi i}\frac{\partial}{\partial j}\log R(j|r_1,r_2)$. The reflection amplitude describes the reflection of an incoming plane wave of spin j coming from spatial infinity, into an outgoing wave with a phase $R(j|r_1,r_2)$; so $R(j|r_1,r_2)$ comes with the pysical requirement that its modulus should be one, expressing the unitarity of the reflection process. This is equivalent to $N(j|r_1, r_2)$ being real.

The definition of the reflection amplitude holds for general quantum mechanical systems living on noncompact spaces, so that we can define asymptotic states and study their reflection properties. In general the density of states suffers from a universal large volume divergence, which can be regularized by considering relative reflection amplitudes and densities of states. In our case this means that we will in fact consider $\frac{R(j|r_1,r_2)}{R(j|0,0)}$ (we will sometimes keep this regularization implicit, as we already did in the above relation between $R(j|r_1, r_2)$ and $N(j|r_1, r_2)$).

The consistency condition deriving from factorization constraints was found in [4, formula (4.43)]

$$\frac{R(j+\frac{1}{2}|r_1,r_2)}{R(j-\frac{1}{2}|r_1,r_2)} = \frac{2j}{2j+1} e_-\left(-j-\frac{1}{2}|r_1,r_2\right), \qquad \forall j \in -\frac{1}{2}+i\mathbb{R}, \tag{1}$$

where we use the function

$$\begin{aligned} e_-(j|r_1,r_2) &= \frac{\Gamma(1+b^2(2j-1))\Gamma(-b^2(2j+1))}{\sin \pi b^2 2j} \times \\ &\quad \times \Pi_{s=\pm} \cos\left(\pi b^2 j + s\frac{i}{2}(r_1+r_2)\right) \sin\left(\pi b^2 j + s\frac{i}{2}(r_1-r_2)\right), \end{aligned} \tag{2}$$

where b^2 is related to the level k by $b^2 = \frac{1}{k-2}$, and we omitted a b-dependent factor. Note that these quantities were originally written in terms of parameters $\rho_{1,2}$ instead of $r_{1,2}$, but solving the case $r_1 = r_2$ was enough to determine ρ as a function of r and b. Moreover, note that the consistency condition (1) involves the analytic continuation of the amplitude $R(j|r_1, r_2)$ outside the physical range. This continuation will give rise to subtleties when $r_1 \neq r_2$.

Now let me solve the equation (1). First recall the solution when $r_1{=}r_2$, already found in [4]:

$$R(P|r,r) = \frac{S_k(\frac{r}{\pi b^2}+P)}{S_k(\frac{r}{\pi b^2}-P)}, \tag{3}$$

here we omit r-independent factors and use the function S_k

$$\log S_k(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\sin 2tb^2x}{2\sinh b^2 t \sinh t} - \frac{x}{t}\right). \tag{4}$$

For $R(j|r, -r)$, let me write an ansatz obtained by replacing S_k with a new function S_k' in eq. (3) :

$$\log S_k'(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\cosh t \sin 2tb^2x}{2\sinh b^2 t \sinh t} - \frac{x}{t}\right). \tag{5}$$

This function $R(P|r,-r) = \frac{S_k'(\frac{r}{\pi b^2}+P)}{S_k'(\frac{r}{\pi b^2}-P)}$ can also be rewritten

$$R(j|r,-r) = \sqrt{\frac{R(j|r+i\frac{\pi}{2}, r+i\frac{\pi}{2})R(j|r-i\frac{\pi}{2}, r-i\frac{\pi}{2})}{R(j|i\frac{\pi}{2}, i\frac{\pi}{2})R(j|-i\frac{\pi}{2}, -i\frac{\pi}{2})}} R(j|0,0). \tag{6}$$

Once we know a reflection amplitude $R(j|r,-r)$ satisfying (1), it is easy to find a solution with arbitrary $r_{1,2}$:

$$R(j|r_1,r_2) = R\left(j|\frac{r_1+r_2}{2},\frac{r_1+r_2}{2}\right) R\left(j|\frac{r_1-r_2}{2},\frac{r_2-r_1}{2}\right) R(j|0,0)^{-1}\,. \tag{7}$$

Our expressions for $R(j|r,-r)$ and $R(j|r_1,r_2)$ are perfectly well-defined and regular on the physical line $j \in -\frac{1}{2}+i\mathbb{R}$. However, in order to prove that our ansatz (6) satisfies eq. (1), we have to define the analytic continuations involved in eq. (1) as well as the squareroot in (6). First notice that the function $S_k(x)$ is analytic in the strip $|\Im x| < \frac{1+b^2}{2b^2}$ and satisfies

$$\frac{S_k(x-\frac{i}{2})}{S_k(x+\frac{i}{2})} = 2\cosh \pi b^2 x\,, \tag{8}$$

so it can be continued to a meromorphic function on the whole complex plane. Similarly, $S'_k(x)$ is analytic in the strip $|\Im P| < \frac{1}{2}$. In order to evaluate eq. (1) we need to go to the boundary of this strip, where we meet singularities. Using eq. (8), $R(P|r,-r)^2$ can easily be defined as a meromorphic function in the whole complex plane, and it satisfies (the square of) eq. (1). However, $R(P|r,-r)$ has branch cuts due to the square root. Indeed, the function $S'_k(x)^2$ has a pole at $x=\frac{i}{2}$ and a zero at $x=-\frac{i}{2}$, as we can see from eq. (8) and the identity

$$S'_k(x)^2 = \frac{S_k(x+\frac{i}{2b^2})S_k(x-\frac{i}{2b^2})}{S_k(\frac{i}{2b^2})S_k(-\frac{i}{2b^2})} S_k(0)^2\,. \tag{9}$$

Branch cuts of $S'_k(x)$ originate at this pole and this zero. Notice that arguments of the reflection amplitude belonging to the physical line $j=-\frac{1}{2}+i\mathbb{R}$ correspond to real values for the argument x of S'_k. By definition (5), the latter function is regular on the real line, thus its branch cuts cannot cross the real line.

The full determination of the branch cuts of S'_k is linked with the interpretation of the factorization constraint (1). Indeed, this constraint should hold in the physical range $j \in -\frac{1}{2}+i\mathbb{R}$, whereas for such j the quantity $R(j+\frac{1}{2}|r_1,r_2)$ needs not even be defined. This comes from the fact that this constraint was obtained from the factorization of an unphysical field $\Phi^{\frac{1}{2}}$. As a result, we need to use an analytical continuation of $R(j|r_1,r_2)$ to unphysical regions, but we have to allow the possibility of meeting branch cuts. However, the r. h. s. of (1) is a meromorphic function on the whole complex plane. So it is natural to define the analytic continuation of $R(j|r_1,r_2)$ outside the physical line by requiring the l. h. s. of (1) to be meromorphic. Equivalently, we require the constraint eq. (1) to hold for all js, not only $j\in -\frac{1}{2}+i\mathbb{R}$. This means in particular that the two branch cuts of S'_k starting at $x=\pm\frac{i}{2}$ have to be correlated, in order for the l. h. s. of (1) not to have any branch cuts. We thus want $S'_k(x+\frac{i}{2})$ and $S'_k(x-\frac{i}{2})$ to hit branch cuts simultaneously, so the branch cuts of $S'_k(x)$ originating at $x=\pm\frac{i}{2}$ should be located either at $x=\pm\frac{i}{2}-\mathbb{R}_+$, or at $x=\pm\frac{i}{2}+\mathbb{R}_+$. Then the function $\frac{R(j+\frac{1}{2}|r,-r)}{R(j-\frac{1}{2}|r,-r)}$ is meromorphic and satisfies eq. (1).

To check these results, one can use them to evaluate the annulus amplitude, and compare it with the similar quantity computed from the boundary states. We now write this computation, but for brevity we neglect all the regularization problems because they have already been dealt with in [4]. This means for instance that we use the character $\chi_P(q) = q^{b^2P^2}/\eta(\tau)^3$ for the continuous representation of spin $j = -\frac{1}{2} + iP$, ignoring the infinite factor coming from the ground state degeneracy. More generally, we will work modulo numerical and τ-dependant factors, since the normalizations have already been fixed in [4]. Also, we will not write explicitly the regularization of the volume divergence which we already mentioned. This regularization consists in using $N(j|r_1, r_2) - N(j|0, 0)$, it allows us to ignore r_1, r_2-independant terms.

We now compute the one-loop partition function of an open string stretching between the two D-branes of parameters r_1 and r_2,

$$
\begin{aligned}
Z_{r_1,r_2}(\tilde{q}) &= \int dP\; N(P|r_1, r_2)\chi_P(q) \\
&= \int dP\, \chi_P(q)\frac{\partial}{\partial P}\left(\log R\left(-\frac{1}{2} + iP|\frac{r_1+r_2}{2}, \frac{r_1+r_2}{2}\right) + \right. \\
&\qquad \left. + \log R\left(-\frac{1}{2} + iP|\frac{r_1-r_2}{2}, -\frac{r_1-r_2}{2}\right)\right) \\
&= \int dP\; \chi_P(q)\frac{\partial}{\partial P}\log\frac{S_k(\frac{r_1+r_2}{2\pi b^2}+P)S'_k(\frac{r_1-r_2}{2\pi b^2}+P)}{S_k(\frac{r_1+r_2}{2\pi b^2}-P)S'_k(\frac{r_1-r_2}{2\pi b^2}-P)} \\
&= \int dP\; \chi_P(q)\int dt\; \frac{\cos 2tb^2P}{\sinh b^2 t \sinh t}\left(\cos\frac{r_1+r_2}{\pi}t + \cosh t\cos\frac{r_1-r_2}{\pi}t\right) \\
&\overset{t=2\pi P'}{=} \int dP'\; \chi_{P'}(\tilde{q})\left[\frac{\cosh^2\pi P'\cos 2r_1P'\cos 2r_2P'}{\sinh 2\pi b^2P'\sinh 2\pi P'} - \right. \qquad (10) \\
&\qquad \left. - \frac{\sinh^2\pi P'\sin 2r_1P'\sin 2r_2P'}{\sinh 2\pi b^2P'\sinh 2\pi P'}\right]. \qquad (11)
\end{aligned}
$$

This should be equal to the closed-string cylinder diagram, computed using the boundary state [4, 5]

$$
\left\langle\Phi^j_{n,p}(z)\right\rangle_r = \frac{e^{-r(2j+1)} + (-1)^n e^{r(2j+1)}}{\Gamma(1+j+\frac{n}{2})\Gamma(1+j-\frac{n}{2})}\Gamma(1+b^2(2j+1))\Gamma(2j+1)\frac{2^{-\frac{3}{4}}b^{-\frac{1}{2}}\nu_b^{j+\frac{1}{2}}\delta(p)}{|z-\bar{z}|^{2\Delta_j}}. \qquad (12)
$$

Here we used the Fourier transform $\Phi^j_{n,p}$ of the basic bulk field $\Phi^j(u)$, defined as $\Phi^j_{n,p} = \int d^2u e^{-in\arg(u)}|u|^{-2j-2-ip}\Phi^j(u)$. Now the cylinder diagram is

$$
Z^{\text{cylinder}}_{r_1,r_2}(\tilde{q}) = \int dP'\; \chi_{P'}(\tilde{q})\sum_{n\in\mathbb{Z}}\int dp\; \left\langle\Phi^{-\frac{1}{2}+iP'}_{n,p}(\frac{i}{2})\right\rangle^*_{r_1}\left\langle\Phi^{-\frac{1}{2}+iP'}_{n,p}(\frac{i}{2})\right\rangle_{r_2}. \qquad (13)
$$

In order to check that $Z^{\text{cylinder}}_{r_1,r_2} = Z_{r_1,r_2}$ holds (modulo the numerical factors and the r_1, r_2-independant terms that we neglect), we would need to compute

$$
\left\langle\Phi^{-\frac{1}{2}+iP'}_{n,p}(\frac{i}{2})\right\rangle^*_{r_1}\left\langle\Phi^{-\frac{1}{2}+iP'}_{n,p}(\frac{i}{2})\right\rangle_{r_2}.
$$

It is easy to see that this quantity depends on n only through the parity of n. More precisely, it corresponds to the term (10) if n is even, and to the term (11) if n is odd.[1]

[1]Of course, in both cases $\sum_n$ gives an infinite result. This can be dealt with by cutting off this sum exactly as in [4].

Let me finally explain why no solution to eq. (1) was found in [4] when $r_1 \neq r_2$. In [4], equation (1) was not used as such but rewritten in the form

$$\left| R\left(j+\frac{1}{2} \,\middle|\, r_1, r_2\right)\right|^2 = \frac{2j}{2j+1} e_-\left(-j-\frac{1}{2}|r_1, r_2\right), \tag{14}$$

which was shown not to admit solutions for $r_1 \neq r_2$. This is simply because the r. h. s. is not a real positive number for all physical values of j if $r_1 \neq r_2$. But the derivation of eq. (14) from eq. (1) implicitly assumes that $R(\bar{j}|r_1, r_2) = \overline{R(j|r_1, r_2)}$. This does not hold for the $R(j|r, -r)$ that we defined, given the locations of the branch cuts, which are not related by $j \to \bar{j}$. A way to see that is to notice that on the branch cuts the argument of the square roots is pure imaginary; and if the function $\sqrt{z}$ has a branch cut $i\mathbb{R}_+$ in the z-plane then it does not satisfy $\overline{\sqrt{z}} = \sqrt{\bar{z}}$.

Acknowledgments

I am very grateful to V. Schomerus and J. Teschner for collaborating on related issues, discussing the present results, and making helpful comments on this manuscript. I thank the AEI, Potdam for hospitality when this research was done.

References

[1] C. Bachas and M. Petropoulos, *Anti-de Sitter D-branes*, *J. High Energy Phys.* **02** (2001) 025 [hep-th/0012234].

[2] P.M. Petropoulos and S. Ribault, *Some remarks on anti-de Sitter D-branes*, *J. High Energy Phys.* **07** (2001) 036 [hep-th/0105252].

[3] P. Lee, H. Ooguri, J. w. Park and J. Tannenhauser, *Open strings on AdS_2 branes*, *Nucl. Phys.* **B 610** (2001) 3 [hep-th/0106129].

[4] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the euclidean AdS_3*, *J. High Energy Phys.* **02** (2002) 016 [hep-th/0112198].

[5] P. Lee, H. Ooguri and J.-w. Park, *Boundary states for AdS_2 branes in AdS_3*, *Nucl. Phys.* **B 632** (2002) 283 [hep-th/0112188].

[6] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and* $\mathrm{SL}(2,\mathbb{R})$ *WZW model. i*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[7] J. Teschner, *Crossing symmetry in the H_3^+ WZNW model*, *Phys. Lett.* **B 521** (2001) 127 [hep-th/0108121].

[8] B. Ponsot, *Monodromy of solutions of the Knizhnik-Zamolodchikov equation:* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 642** (2002) 114 [hep-th/0204085].

[9] S. Ribault and V. Schomerus, work in progress.